# Adapted operator representations: Selective versus collective properties of quantum networks

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Based on local unitary operators acting on a $n$-dimensional Hilbert-space, we investigate selective and collective operator basis sets for $N$-particle quantum networks. Selective cluster operators are used to derive the properties of general cat-states for any $n$ and $N$. Collective operators are conveniently used to account for permutation symmetry: The respective Hilbert-space dimension is then only polynomial in $N$ and governed by strong selection rules. These selection rules can be exploited for the design of decoherence-free subspaces as well as for the implementation of efficient routes to entanglement if suspended switching between states of different symmetry classes could be realized.

## I. INTRODUCTION

Problem-adapted representations are convenient tools for dealing with concrete physical models in virtually any branch of physics: Examples are the choice of coordinates in classical mechanics, of mode representations in linear wave theory, of state vector- or matrix-representations and of complete operator basis sets in quantum mechanics. The mode of adaption may refer to the internal symmetry of the system under consideration and/or its coupling to the outside world (means of measurement and control). Adapted representations - though in principle equivalent to any other - are expected to simplify numerical calculations and to enhance insight. (cf. [1])

Operators in quantum mechanics may represent observables, states or transformations. In any case it is convenient to think in terms of "elements", i.e. basic operator sets, out of which any other operator could be constructed [2]: For a $n$-dimensional Hilbert-space there are $n^2$ such (orthogonal) basis operators (defining Liouville-space). If we prefer to think in terms of basic observables or states, these basis operators should be chosen hermitian; a pertinent example are the $SU(n)$-generators. If we rather think in terms of basic "actions", the basis-operators should be unitary (thus defining basic unitary transformation). The latter approach has become the method of choice for investigations relating to quantum computation and quantum information processing [3]: there we are typically concerned with sequences of unitary transformations. (Only for $n = 2$ are unitarity and hermiticity compatible requirements for a complete basis set.)

Quantum networks (composite systems) may be described in terms of product-operators, $\hat{Q}$; if each local operator is taken from the respective unitary basis set, also the product-operators are unitary. Furthermore, they are completely specified by the type of "action" to be applied on each subsystem $\mu$, $\mu = 1, 2, \ldots, N$; these $\hat{Q}$-operators will be termed "selective". Alternatively, we may introduce operators $\hat{E}$ which specify the action but not the "address". In this case we are naturally led to "collective operators", defining a specific action on a given number $\alpha$ of subsystems. Complete sets require the inclusion of phases. Collective as well as selective operator sets are equivalent: in particular, we can express one type by the other. A subset of collective operators has permutation symmetry: These are the only allowed operators for fundamentally indistinguishable subystems (fermions or bosons).

Typical scenarios realized, e.g. in nanostructures, will exhibit neither complete selectivity nor complete non-selectivity (permutation symmetry). However, systems with weakly restricted selectivity should still more efficiently be described by $\hat{Q}$-Operators, systems with weakly broken permutation symmetry by collective operators. It is this latter theme of operational (partial) indistinguishability which will be of central interest in our present investigation.

Our paper is organized as follows: In section II we discuss local basis operators with special emphasis on unitary operators and introduce the complex coherence vector as a generalization to the Bloch vector. Section III extends this concept to quantum networks by the use of index-selective cluster operators. Alternatively, section IV proposes the use of collective operators for which all subsystems are treated on equal footing. Applications of these concepts are worked out in section V, starting with highly entangled states and generalized cat states. Permutation symmetry plays the key role in the remaining applications showing the fundamental difference between selective and collective treatment of quantum networks.

## II. BASIS OPERATORS

### A. Transition- and SU($n$)-operators

The Hilbert-space of dimension $n$ is taken to be spanned by a complete and orthonormal set of states $|j\rangle$, $j = 0, 1, 2, \ldots, n-1$. They may be considered eigenstates

of some operator $\hat{A}$,

$$\hat{A} = \sum_{j=0}^{n-1} A_j \hat{P}_{jj} \tag{2.1}$$

where $\hat{P}_{ij} = |i\rangle\langle j|$; $\hat{P}_{ij}^\dagger = \hat{P}_{ji}$ denote $n^2$ transition operators, which are characterized by two state indices (= quantum numbers = eigenvalues $A_j$); they are orthonormalized, $\mathrm{tr}\left\{\hat{P}_{ij}\hat{P}_{kl}^\dagger\right\} = \delta_{ik}\delta_{jl}$, (tr = trace operation) and complete. Any other operator $\hat{B}_s$ can then be written as

$$\hat{B}_s = \sum_{ij} B_{s,ij}\hat{P}_{ij} \tag{2.2}$$

where $B_{s,ij} = \mathrm{tr}\left\{\hat{B}_s\hat{P}_{ij}^\dagger\right\}$. We require the $\hat{B}_s$, $(s = 0, 1, \ldots, n^2-1)$ to form a complete orthogonal set, normalized to $n$,

$$\mathrm{tr}\left\{\hat{B}_s\hat{B}_{s'}^\dagger\right\} = \sum_{ij} B_{s,ij}B_{s',ij}^* = n\,\delta_{ss'}\ . \tag{2.3}$$

(Remember that the identity operator $\hat{1}$ is also normalized to $n$.) Note, that the transformation eq. (2.2) does not change the representation of any given operator; rather we change the set of operators *within* the $\hat{A}$-representation (i.e. the set of elementary matrices). A well-known example are the $n^2$ hermitian SU($n$)-generators $\underline{\hat{\lambda}} = \{\hat{\lambda}_s, s = 0, 1, \ldots, n^2-1\} = \{\hat{1}, \hat{u}_{01}, \hat{u}_{02}, \ldots, \hat{u}_{12}, \ldots, \hat{v}_{01}, \ldots, \hat{w}_0, \ldots, \hat{w}_{n-2}\}$

$$\hat{\lambda}_0 \equiv \hat{1} = \sum_{j=0}^{n-1}\hat{P}_{jj}\ , \tag{2.4}$$

$$\hat{u}_{ik} = \sqrt{\frac{n}{2}}(\hat{P}_{ik} + \hat{P}_{ki})\ , \tag{2.5}$$

$$\hat{v}_{ik} = \sqrt{\frac{n}{2}}i(\hat{P}_{ik} - \hat{P}_{ki})\ , \tag{2.6}$$

$$\hat{w}_l = -\sqrt{\frac{n}{(l+1)(l+2)}}(\hat{P}_{00} + \ldots + \hat{P}_{ll} - (l+1)\hat{P}_{l+1,l+1})\ , \tag{2.7}$$

with $\mathrm{tr}\left\{\hat{\lambda}_s\hat{\lambda}_{s'}\right\} = n\,\delta_{s,s'}$ for all $s$, $s'$. (The SU($n$)-generators are often normalized to 2 rather than $n$ [2]; the latter choice is more convenient for our purposes.) For $n = 2$ the representation of the $\hat{\lambda}_j = \hat{\lambda}_0, \hat{u}_{01}, \hat{v}_{01}, \hat{w}_0$ leads to the well-known Pauli matrices. For $n > 2$ the SU($n$)-algebra tends to loose its convenience, because the corresponding operators are no longer unitary and their definition becomes rather unwieldy (cf. [4]).

With the unitary operators to be discussed next, the SU($n$)-operators share two properties:

i. The set consists of $n^2-1$ traceless operators $\hat{B}_s$, $s \neq 0$ and the unit-operator $\hat{B}_0 = \hat{1}$.

ii. The operators based on projection operators $\hat{P}_{jj}$ are kept separated from the others.

As a result of i., the product-operators of a composite system (cf. Sect. III,IV can be decomposed into a hierachy of $m$-cluster-operators $\hat{Q}$, where $m$ is the number of indices $s \neq 0$.

As a result of ii., and if $\hat{A} = \hat{H}$, the Hamiltonian of the system under consideration, the expectation values of the $\hat{P}_{jj}$ or combinations of those are constants of motion (under the unitary evolution generated by $\hat{H}$).

### B. Unitary operators

The operators [5,6]

$$\hat{U}_{ab} := \sum_{k=0}^{n-1}\omega_n^{bk}\,|\underline{k+a}\rangle\langle k|\ ; \qquad \omega_n = e^{\frac{2\pi i}{n}} \tag{2.8}$$

are defined in a double index notation $a, b \in [0, n-1]$, where addition modulo $n$ is denoted by underlining and the constant $\omega_n$ is the $n$-th root of unity. We note for later reference, that

$$\sum_{k=0}^{n-1}\omega_n^{bk} = n\delta_{0b}\ ; \quad \sum_{b=0}^{n-1}\omega_n^{b(k-k_0)} = n\delta_{kk_0}\ , \tag{2.9}$$

where $\delta_{ij}$ is the Kronecker-delta. The use of two indices leads to an easy interpretation of the action such an operator has on a state $|k\rangle$,

$$\hat{U}_{ab}\,|k\rangle = \omega_n^{bk}\,|\underline{k+a}\rangle\ , \tag{2.10}$$

namely $b$ induces a phase shift and $a$ causes a state-shift (for a concrete example see Appendix A). This set of $n^2$ operators defines an orthonormal and complete operator basis for the Liouville-space with

$$\mathrm{tr}\left\{\hat{U}_{ab}\hat{U}_{cd}^\dagger\right\} = n\delta_{ac}\delta_{bd}\ . \tag{2.11}$$

A operator $\hat{A}$ can be expanded like

$$\hat{A} = \frac{1}{n}\sum_{a,b=0}^{n-1}u_{ab}\hat{U}_{ab}\ ; \qquad u_{ab} := \mathrm{tr}\left\{\hat{U}_{ab}^\dagger\hat{A}\right\}\ . \tag{2.12}$$

Coming next we will state some of the basic properties these unitary operators obey. Because $\hat{U}_{00} = \hat{1}$, the orthonormality relation (2.11) implies $\mathrm{tr}\left\{\hat{U}_{ab}\right\} = 0$ for $a, b \neq 0$. This tracelessness will be essential for the definition of cluster-operators (c.f. Sect. III). The adjoint operators are again, barring a phase, members of the set,

$$\hat{U}_{ab}^\dagger = \hat{U}_{ab}^{-1} = \omega_n^{ab}\hat{U}_{\underline{-a},\underline{-b}}\ . \tag{2.13}$$

A very useful property of these operators is their cyclic symmetry, implying that any product reduces to just one operator of the set:

$$\hat{U}_{ab}\hat{U}_{cd} = \omega_n^{bc}\hat{U}_{\underline{a+c},\underline{b+d}} \tag{2.14}$$

$$\hat{U}_{ab}\hat{U}_{cd}^\dagger = \omega_n^{cd}\omega_n^{-bc}\hat{U}_{\underline{a-c},\underline{b-d}} \tag{2.15}$$

$$\hat{U}_{cd}^\dagger\hat{U}_{ab} = \omega_n^{cd}\omega_n^{-ad}\hat{U}_{\underline{a-c},\underline{b-d}} \tag{2.16}$$

$$\hat{U}_{ab}\hat{U}_{cd}\hat{U}_{ef} = \omega_n^{bc}\omega_n^{(b+d)e}\hat{U}_{\underline{a+c+e},\underline{b+d+f}} \tag{2.17}$$

It is remarkable to note that the cyclic property allows to use - instead of the complete set $\{\hat{U}_{ab}\}$ - the two operators $\{\hat{U}_{0,n-1}, \hat{U}_{n-1,0}\}$ only. All the others can then be generated as specific product forms, e.g. for $n = 3$: $\hat{U}_{00} = (\hat{U}_{20})^3, \hat{U}_{10} = (\hat{U}_{20})^2, \hat{U}_{22} = \hat{U}_{20}\cdot\hat{U}_{02}, \hat{U}_{01} = (\hat{U}_{02})^2, \hat{U}_{12} = (\hat{U}_{20})^2\cdot\hat{U}_{02}$, etc. This property is reminescent of the creation- and destruction operators, $\hat{a}^\dagger, \hat{a}$, conveniently applied to harmonic oscillator models; it introduces a kind of "non-linearity" if expectation values of products (cf. eq. (2.28)) are approximated by products of expectation values.

From eq. (2.17) and eq. (2.13) we conclude that

$$\hat{U}_{0d}^\dagger\cdot\hat{U}_{ab}\cdot\hat{U}_{0d} = \omega_n^{-da}\hat{U}_{ab} \tag{2.18}$$

$$\hat{U}_{d0}^\dagger\cdot\hat{U}_{ab}\cdot\hat{U}_{d0} = \omega_n^{bd}\hat{U}_{ab} \tag{2.19}$$

i.e. $\hat{U}_{a0}$ (and their combinations) are invariant under cyclic permutations of states, the $\hat{U}_{0a}$ are invariant under cyclic permutations of phases.

The determinant of $\hat{U}_{ab}$ can only take on the values $\pm 1$,

$$\det\hat{U}_{ab} = (-1)^{(a+b)(n-1)}\,, \tag{2.20}$$

and the eigenvalues, consecutively numbered by $k$, all lie on the unit circle in the complex plane

$$\lambda_k^n = \omega_n^{bl+ab\frac{n(n-1)}{2}} = (-1)^{ab(n-1)}\omega_n^{bl}\,. \tag{2.21}$$

From the cyclic symmetry the commutation properties are found to be

$$\left[\hat{U}_{ab},\hat{U}_{cd}\right]_\pm = (\omega_n^{bc}\pm\omega_n^{ad})\hat{U}_{\underline{a+c},\underline{b+d}}\,, \tag{2.22}$$

$$\left[\hat{U}_{ab},\hat{U}_{cd}^\dagger\right]_\pm = \omega_n^{cd}(\omega_n^{-bc}\pm\omega_n^{-ad})\hat{U}_{\underline{a-c},\underline{b-d}}\,, \tag{2.23}$$

and the structure constants $f_{ab,cd,ef}$, defined by

$$\left[\hat{U}_{ab}^\dagger,\hat{U}_{cd}\right]_- = \sum_{e,f=0}^{n-1} f_{ab,cd,ef}\,\hat{U}_{ef}\,, \tag{2.24}$$

follow as

$$f_{ab,cd,ef} := \omega_n^{ab}\left(\omega_n^{-bc}-\omega_n^{-ad}\right)\delta_{e,\underline{a+c}}\delta_{f,\underline{b+d}}\,. \tag{2.25}$$

These structure constants are much simpler than those for SU($n$)! Relations to other basis sets are summarized in Appendix B.

### C. Complex coherence vector

The general state of a quantum-mechanical system is specified by its density-operator $\hat{\rho} = \sum_{i,j}\rho_{ij}\hat{P}_{ij}$ with $\rho_{ij} = \mathrm{tr}\left\{\hat{\rho}\hat{P}_{ij}^\dagger\right\}$ defining the respective density matrix. In many areas of physics, especially for finite-dimensional state spaces, the description of $\hat{\rho}$ using a coherence vector (as the set of expectation values of the underlying SU($n$)-operator basis) has shown great power because of its intuitive, almost "classical" behaviour. For two-level systems the coherence vector lives in an ordinary 3-dimensional space. In general, the SU($n$) basis, which is hermitian, leads to a $(n^2-1)$-dimensional vector consisting of real values, while the unitary operators studied here give complex elements.

For the operators $\hat{U}_{ab}$ we will henceforth use a single index notation $\hat{U}_i$ with $i\in[0,n^2-1]$, interpreting $ab$ as the $n$-adic number representing $i$.

Following eq. (2.12), we can expand the density operator $\hat{\rho}$ as

$$\hat{\rho} = \frac{1}{n}\sum_{i=0}^{n^2-1} u_i\hat{U}_i \tag{2.26}$$

and collect the coefficients to define the complex coherence vector

$$\mathbf{u} := \{u_i \quad , \quad 0<i<n^2-1\} \tag{2.27}$$

$$u_i := \left\langle\hat{U}_i^\dagger\right\rangle = \mathrm{tr}\{\hat{U}_i^\dagger\hat{\rho}\} \quad \in\mathbb{C} \tag{2.28}$$

where we deliberately excluded $u_0$ since it always equals 1 because of $\hat{U}_0 = \hat{1}$. From the symmetry (cf. eq. (2.13))

$$u_{ab} = u^*_{\underline{-a},\underline{-b}}\,\omega_n^{ab}\,, \tag{2.29}$$

we conclude that there are $n^2-1$ independent real parameters forming the complex coherence vector. The length of $\mathbf{u}$ can be related to the trace of the density operator $\hat{\rho}$

$$|\mathbf{u}|^2 = n\,\mathrm{tr}\{\hat{\rho}^2\} - 1\,, \tag{2.30}$$

thus giving a simple criterion to distinguish between pure ($\mathrm{tr}\{\hat{\rho}^2\}=1$) and mixed ($\mathrm{tr}\{\hat{\rho}^2\}<1$) states:

$$|\mathbf{u}|^2 = n-1 \qquad \text{pure state} \tag{2.31}$$

$$0\le|\mathbf{u}|^2 < n-1 \qquad \text{mixed state} \tag{2.32}$$

The convenience of the well known Bloch vector formalism comes from the simple motion performed by the vector under unitary transformations: Since the length

$|\mathbf{u}|$ is preserved, any unitary time evolution operator $\hat{U}(t)$ just causes a rotation of the coherence vector. If the density matrix evolves like $\hat{\rho}(t) = \hat{U}(t)\hat{\rho}(0)\hat{U}(t)^\dagger$, one can easily show that the motion of $\mathbf{u}$ is

$$\mathbf{u(t)} = \mathrm{T(t)}\ \mathbf{u(0)}\ , \tag{2.33}$$

with rotation matrix $\mathrm{T}(t)$ [7],

$$T_{ij}(t) := \frac{1}{n}\,\mathrm{tr}\{\hat{U}_j^\dagger \hat{U}(t)^\dagger \hat{U}_i^\dagger \hat{U}(t)\}\ . \tag{2.34}$$

Using the Liouville equation one can replace the time evolution operator by the Hamiltonian $\hat{H}(t)$ leading to a differential equation similar to the Bloch equations,

$$\dot{\mathbf{u}} = \Omega\ \mathbf{u}\ ; \qquad \Omega_{ij} := -\frac{1}{ni\hbar}\,\mathrm{tr}\{\hat{H}(t)\ \left[\hat{U}_i^\dagger, \hat{U}_j\right]_-\}\ . \tag{2.35}$$

As expected, $\Omega$ with its properties $\Omega_{ij} = -\,\Omega_{ji}^*$ and $\mathrm{tr}\{\Omega_{ij}\} = 0$ describes a rotation in complex vector-space.

## III. CLUSTER-OPERATOR BASIS

### A. Definition and properties

Up to now we have restricted ourselves to a single system, respective one node of a quantum network. To describe a network of $N$ subsystems, $N > 1$, looking at each node separately is not enough since correlations between nodes emerge – so the concept of clusters has to be introduced. A cluster operator acting on $m$ particles is build out of $m$ one particle operators $\hat{U}_i^{(\mu)}$, $i \neq 0$, where greek indices label different nodes of the network. All cluster operators

$$\hat{U}_i^{(\mu)} := \hat{1} \otimes \ldots \otimes \hat{1} \otimes \underbrace{\hat{U}_i}_{\text{node } \mu} \otimes \hat{1} \otimes \ldots \otimes \hat{1}\ , \tag{3.1}$$

$$\hat{U}_{ij}^{(\mu\nu)} := \hat{1} \otimes \ldots \underbrace{\hat{U}_i}_{\text{node } \mu} \ldots \otimes \hat{1} \otimes \ldots \underbrace{\hat{U}_j}_{\text{node } \nu} \ldots \otimes \hat{1} \quad \text{etc.} \tag{3.2}$$

are unitary and orthonormal, and together span the complete Liouville-space of the quantum network. Acting on only $m$ nodes out of $N$ means leaving the others unaffected by choosing $\hat{U}_0 = \hat{1}$ for them (Thus $m$ is the number of indices $i, j, \ldots$ unequal zero). The general decomposition of $\hat{\rho}$ (like of any other operator $\hat{A}$) for a $N$ particle quantum network with node $\nu$ being a $n_\nu$-level system reads

$$\hat{\rho} = \frac{1}{\prod_{\mu=1}^{N} n_\mu}\left(\hat{1} + \sum_{\mu=1}^{N}\sum_{i=1}^{n_\mu^2-1} u_i^{(\mu)}\hat{U}_i^{(\mu)} + \sum_{\mu<\nu}\sum_{i,j} u_{ij}^{(\mu\nu)}\hat{U}_{ij}^{(\mu\nu)} + \sum_{\mu<\nu<\sigma}\sum_{i,j,k} u_{ijk}^{(\mu\nu\sigma)}\hat{U}_{ijk}^{(\mu\nu\sigma)} + \ldots + \sum_{\underbrace{i,j,k,\ldots,l}_{N \text{ indices}}} u_{ijk\ldots l}^{(12\ldots N)}\hat{U}_{ijk\ldots l}^{(12\ldots N)}\right)\ , \tag{3.3}$$

with the index-selective expectation values

$$u_{ij\ldots k}^{(\mu\nu\ldots\sigma)} := \mathrm{tr}\left\{\hat{\rho}\cdot \hat{U}_{ij\ldots k}^{(\mu\nu\ldots\sigma)^\dagger}\right\} = \left\langle \hat{U}_{ij\ldots k}^{(\mu\nu\ldots\sigma)^\dagger}\right\rangle\ . \tag{3.4}$$

The local coherence vectors discussed in Sect. II B just turn out to be the correlation tensors of first order, $m = 1$. If the network is in a product state, all correlation tensors of higher order factor into a product of local coherence vectors,

$$u_{ij\ldots k}^{(\mu\nu\ldots\sigma)} = u_i^{(\mu)} u_j^{(\nu)} \ldots u_k^{(\sigma)} \tag{3.5}$$

i.e. these states are completely determined by local properties. In case of a single subsystem, the length of its coherence vector $\sum_i |u_i|^2$ can be identified with a scalar constant of motion under unitary evaluation. For higher order correlation tensors, the concept of cluster sums leads to new invariants. For any $m$ particle cluster, the cluster sum is defined as the weight of the tensor (cf. [2])

$$Y_m^{(\mu\nu\ldots\sigma)} := \sum_{\underbrace{i,j,\ldots,k}_{m \text{ indices}\neq 0}} \left|u_{ij\ldots k}^{(\mu\nu\ldots\sigma)}\right|^2\ , \tag{3.6}$$

with special cases $Y_0 = 1$ and $Y_1^{(\mu)} = \left|\mathbf{u}^{(\mu)}\right|^2$. For a network with $N$ nodes, these cluster sums give $2^N$ scalar invariants under (products of) local unitary transformations. All the cluster sums together are subject to the sum rule

$$\mathrm{tr}\left\{\hat{\rho}^2\right\}\prod_{\mu=1}^{N} n_\mu = Y_0 + \sum_{\mu=1}^{N} Y_1^{(\mu)} + \sum_{\mu<\nu} Y_2^{(\mu\nu)} + \sum_{\mu<\nu<\sigma} Y_3^{(\mu\nu\sigma)} + \ldots + Y_N^{(123\ldots N)}\ . \tag{3.7}$$

For a pure state, $\mathrm{tr}\left\{\hat{\rho}^2\right\} = 1$ and for $n_\mu = n$ the left hand side is $n^N$. According to eq. (3.5), any $m$-cluster sum of a product state factorizes into its 1-cluster-sum components. A given cluster of size $m$ can then be tested as being in a non-product-state, if there is some partition into smaller clusters, the cluster-sum product of which is smaller than $Y_m$, e.g. $Y_1^{(1)}Y_1^{(2)} < Y_2^{(12)}$. The cluster

sums can be related to the "purity" of a $m$ particle cluster by interpreting the cluster as a $n^m$-level system with coherence vector $\mathbf{u_m}$. (For simplicity we assume $n_\mu = n$.) A normalized "purity factor" can now be defined from the respective coherence vector length as ($m \leq N$)

$$p_m := \frac{|\mathbf{u_m}|^2}{|\mathbf{u_m}|^2_{\max}} = \frac{n^m \mathrm{tr}\left\{\hat{\rho}^2_{\mathrm{Cluster}}\right\} - 1}{n^m - 1} \tag{3.8}$$
$$= \frac{1}{n^m - 1}\left(\sum_{\mu=1}^{m} Y_1^{(\mu)} + \sum_{\mu<\nu} Y_2^{(\mu\nu)} + \ldots + Y_m^{(12\ldots m)}\right) ,$$

characterizing the purity of a $m$-node cluster on the scale (cf. eq. (2.32))

$$\begin{aligned} 0 \leq p_m \leq 1 & \qquad (3.9)\\ p_m = 0 \quad &\Longleftrightarrow \quad \text{maximal mixed } m\text{-cluster} \\ p_m = 1 \quad &\Longleftrightarrow \quad \text{pure } m\text{-cluster.} \end{aligned}$$

Alternatively, the purity could be characterized by the respective cluster-entropy [8]; $p_m$ has the advantage of being a simple algebraic function of the expectation values (matrix-elements of the reduced density matrix).

## IV. COLLECTIVE OPERATOR-BASIS FOR TWO-LEVEL SUBSYSTEMS

### A. Definition and properties

For now we will stick to quantum networks build out of $N$ two level systems, knowing that a generalization to $n$-level systems is straight forward. First one needs to specify which single particle operator basis is used, e.g. $\hat{u}_{01} = \hat{\sigma}_x$, $\hat{v}_{01} = \hat{\sigma}_y$, $\hat{w}_0 = \hat{\sigma}_z$ (which is hermitian and unitary) or $\hat{\sigma}_\pm = \hat{\sigma}_x \pm i\hat{\sigma}_y$, $\hat{\sigma}_z$ or any other complete basis. We then relabel the cluster operators $\hat{U}^{(12\ldots N)}_{ij\ldots k}$ as $\hat{C}_{\alpha\beta\gamma,p}$, meaning an operator of dimension $2^{2N}$, where $\alpha$,$\beta$ and $\gamma$ specify the multiplicity of $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$, respectively ($\alpha + \beta + \gamma \leq N$). Index $p$ specifies a permutation of these among the $N$ subsystems. The number of such permutations and hence the index range for $p \in [0, \Omega - 1]$ is

$$\Omega(\alpha, \beta, \gamma) = \frac{N!}{\alpha!\beta!\gamma!(N - \alpha - \beta - \gamma)!} . \tag{4.1}$$

These operators $\hat{C}_{\alpha\beta\gamma,p}$, again, span the whole Liouville-space by defining all subsystem specific properties. We now go on to define collective operators $\hat{E}$ by

$$\hat{E}_{\alpha\beta\gamma,b} = \sum_{p=0}^{\Omega-1} \omega_\Omega^{pb} \hat{C}_{\alpha\beta\gamma,p} ; \qquad \omega_\Omega = e^{\frac{2\pi i}{\Omega}} . \tag{4.2}$$

To ensure that all subsystems are treated on equal footing, the sum extends over all permutations $p$, weighted only with pure phase fators, where $b \in [0, \Omega - 1]$ labels the phase shift between "neighbouring" $p$. (Here the numbering of permutations is a matter of choice and the phase has no physical meaning.) The set of collective operators is orthonormal

$$\frac{1}{\Omega\, 2^N} \mathrm{tr}\left\{\hat{E}_{\alpha\beta\gamma,b}\hat{E}^\dagger_{\alpha'\beta'\gamma',b'}\right\} = \delta_{\alpha\alpha'}\delta_{\beta\beta'}\delta_{\gamma\gamma'}\delta_{bb'} \tag{4.3}$$

and complete, so that the density operator $\hat{\rho}$ of the network can be decomposed as

$$\hat{\rho} = \frac{1}{2^N} \sum_{\{\alpha\beta\gamma\}} \sum_b E_{\alpha\beta\gamma,b}\hat{E}_{\alpha\beta\gamma,b} , \tag{4.4}$$

$$E_{\alpha\beta\gamma,b} = \frac{1}{\Omega} \mathrm{tr}\left\{\hat{\rho} \cdot \hat{E}^\dagger_{\alpha\beta\gamma,b}\right\} . \tag{4.5}$$

The expectation values $E_{\alpha\beta\gamma,b}$ are collective in the sense that they do *not* refer to specific subsystem-indices (as opposed to the $\hat{U}$-operators, cf. eq. (3.1,3.2)). The inverse transformation is given by

$$\sum_{b=0}^{\Omega-1} \omega_\Omega^{-bp_0} \hat{E}_{\alpha\beta\gamma,b} = \sum_{p=0}^{\Omega-1} \hat{C}_{\alpha\beta\gamma,b} \sum_{b=0}^{\Omega-1} \omega_\Omega^{(p-p_0)b} = \Omega \hat{C}_{\alpha\beta\gamma,p_0} , \tag{4.6}$$

where we have made use of eq. (2.9). For example, the selective operator $\hat{C}_{100,0} \equiv \hat{\lambda}^{(1)}_1 \otimes \hat{1} \otimes \ldots$ could be written as $\frac{1}{\Omega}\sum_b^{\Omega-1} \hat{E}_{100,b}$. Note that the operators $\hat{E}$ are, in general, no longer unitary; for $b = 0$ they are hermitian if the operators $\hat{C}_{\alpha\beta\gamma,p}$ are.

### B. Alternative sets

The operators $\hat{E}$ still distinguish between all three basic operators. Further reductions are possible: one such variant is (based on $\hat{\sigma}_+$, $\hat{\sigma}_-$, $\hat{\sigma}_z$)

$$\hat{F}_{z,\gamma,b} = \sum_{p=0}^{\Omega(z,\gamma)} \sum_{\alpha,\beta}^{\alpha-\beta=z} \omega^{bp}_{\Omega(z,\gamma)} \hat{C}_{\alpha\beta\gamma,p} \tag{4.7}$$

which has been considered for nuclear spin-networks. In NMR the $\hat{F}_z$-terms in a Hamiltonian are said to induce a total of $z$ quantum-"flip-flops" [9]; an $\hat{F}_z$-term entering the density-operator $\hat{\rho}$ describes the respective coherence order $|z|$.

Another possibility is

$$\hat{G}_{m,b} = \sum_{p=0}^{\Omega(m)} \sum_{\alpha,\beta,\gamma}^{\alpha+\beta+\gamma=m} \omega^{bp}_{\Omega(m)} \hat{C}_{\alpha\beta\gamma,p} \tag{4.8}$$

where $p$ are permutations of the $m$ operators (of any type) on the $N$ subsystems, $m = 0, 1, \ldots, N$. Also this set is still complete.

## V. APPLICATIONS

### A. Commuting operator sets and generalized cat basis

#### 1. Commuting sets of cluster operators

Cluster sums and purity factors can be used to classify states by their non-local properties. Highly entangled states tend to share correlations among all nodes rather than between only a few. Bell states [10] ($N = 2$) are perfect in this sense, because they are in a totally mixed state ($Y_1 = p_1 = 0$) locally and pure otherwise ($Y_2 = 3, p_2 = 1$). Generalizing this point, one may define "highly entangled states" as states that have a cluster sum distribution with a strong focus on multi-particle correlations. This definition does not give us a quantitative measure for multi-particle entanglement, but it can assist us in the search for new states with correlations of high order. Using the unitary operators defined above, we can give a constructive way of finding such states. We restrict ourselves to $n_\mu = n$.

*Lemma 1*
Let $|\psi\rangle$ be an eigenstate of $n^N$ completely commuting unitary operators out of which there are $q_N$ $N$-cluster-opererators. Then its highest order cluster sum is given by $Y_N = q_N$.

Note that the modulus of any eigenvalue of any cluster-operator $U_{ij\ldots}^{(\mu\nu\ldots)}$ is exactly 1; the $q_N$ commuting $N$-cluster-operators thus imply $Y_N = q_N$ provided all the non-commuting $N$-cluster-operators contribute zero. This must be the case, however, if there is a total of $n^N$ commuting operators exploiting the sum rule eq. (3.7).

A set of completely commuting operators is a set in which each operator commutes with all others. This lemma reduces the problem of finding a state with maximum cluster sum $Y_N$ to the problem of finding a maximum set of commuting $N$-cluster-operators. Before we can make statements about cluster operators, the commuting relations of single particle unitary operators $\hat{U}_i$ need to be examined. Since $\left[\hat{U}_{ab}^\dagger, \hat{U}_{cd}\right]_- = 0$ iff $\underline{ad - bc} = 0$, one can show:

*Lemma 2. (based on theory of congruency classes)*
Each operator $\hat{U}_{ab}$ commutes with $n\, gcd(a, b, n)$ other basis operators and to each operator $\hat{U}_{ab}$ there exists a trivial set of $n$ completely commuting basis operators. (gcd = greatest common divisor)

One of the $n$ commuting operators in the completely commuting set is $\hat{U}_0 = \hat{1}$, so the maximal number of commuting one particle operators is $n - 1$.

For a network of $N$ particles with $n$-levels each, the properties of unitary cluster operators lead to the commutator relation

$$\left[\hat{U}_{a_1b_1} \otimes \hat{U}_{a_2b_2} \otimes ..., \hat{U}_{c_1d_1} \otimes \hat{U}_{c_2d_2} \otimes ...\right]_- = 0 \Longleftrightarrow$$
$$\underline{a_1d_1 - b_1c_1 + a_2d_2 - b_2c_2 + ...} = 0\,, \quad (5.1)$$

and from this we find the following statements:

A There exists a set of $(n-1)^N$ completely commuting cluster operators of size $N$. This set can be constructed from all combinations of $n-1$ completely commuting one particle cluster operators.

B There is a set of

$$(n^2-1)^{\frac{N}{2}}, \qquad N \text{ even} \quad (5.2)$$
$$(n^2-1)^{\frac{N-1}{2}}\,(n-1), \qquad N \text{ odd} \quad (5.3)$$

completely commuting $N$ particle cluster operators of the form

$$\hat{U}_{a_1b_1} \otimes \hat{U}_{b_1a_1} \otimes \hat{U}_{a_2b_2} \otimes \hat{U}_{b_2a_2} \ldots\,. \quad (5.4)$$

C There can be larger sets of completely commuting operators than given by eq. (5.3). However, no general constructive method to find them is known to us so far. For $n > 2$ we restrict ourselves to $N \leq 3$.

D The maximum order of any such set is constrained by $n^N - 1$. This maximum value is typically not reached for $N > 2$.

The results of A, B, C and D are summarized in Table I.

$n = 2$

| $N$ | A | B | C | D | Cat |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 3 | 3 | 3 | 3 |
| 3 | 1 | 3 | 4 | 7 | 4 |
| 4 | 1 | 9 | 9 | 15 | 9 |
| 5 | 1 | 9 | 16 | 31 | 16 |
| 6 | 1 | 27 | 33 | 63 | 33 |

$n = 3$

| $N$ | A | B | C | D | Cat |
|---|---|---|---|---|---|
| 1 | 2 | 2 | 2 | 2 | 2 |
| 2 | 4 | 8 | 8 | 8 | 8 |
| 3 | 8 | 16 | 20 | 26 | 20 |
| 4 | 16 | 64 | ? | 80 | 60 |
| 5 | 32 | 128 | ? | 242 | 172 |
| 6 | 64 | 512 | ? | 728 | 508 |

$n = 4$

| $N$ | A | B | C | D | Cat |
|---|---|---|---|---|---|
| 1 | 3 | 3 | 3 | 3 | 3 |
| 2 | 9 | 15 | 15 | 15 | 15 |
| 3 | 27 | 45 | 54 | 63 | 54 |
| 4 | 81 | 175 | ? | 255 | 213 |
| 5 | 243 | 525 | ? | 1023 | 828 |

TABLE I. Cluster sums $Y_N$ for eigenstates corresponding to sets of completely commuting $N$ particle cluster operators calculated by methods A, B and C. Column D gives the upper limit and colum Cat states the cluster sum $Y_N$ for generalized cat states as introduced in Sect. V A 2.

*2. Generalized cat states*

As a first example let us look at the well known case of two spin 1/2 particles. The maximum order for completely commuting sets of 2-cluster-operators is 3 and there are 6 such sets:

$$\begin{aligned}
&\left\{\hat{U}_{01}\otimes\hat{U}_{01},\ \hat{U}_{10}\otimes\hat{U}_{10},\ \hat{U}_{11}\otimes\hat{U}_{11}\right\}\ , \qquad (5.5)\\
&\left\{\hat{U}_{01}\otimes\hat{U}_{01},\ \hat{U}_{10}\otimes\hat{U}_{11},\ \hat{U}_{11}\otimes\hat{U}_{10}\right\}\ ,\\
&\left\{\hat{U}_{01}\otimes\hat{U}_{10},\ \hat{U}_{10}\otimes\hat{U}_{01},\ \hat{U}_{11}\otimes\hat{U}_{11}\right\}\ ,\\
&\left\{\hat{U}_{01}\otimes\hat{U}_{10},\ \hat{U}_{10}\otimes\hat{U}_{11},\ \hat{U}_{11}\otimes\hat{U}_{01}\right\}\ ,\\
&\left\{\hat{U}_{01}\otimes\hat{U}_{11},\ \hat{U}_{10}\otimes\hat{U}_{01},\ \hat{U}_{11}\otimes\hat{U}_{10}\right\}\ ,\\
&\left\{\hat{U}_{01}\otimes\hat{U}_{11},\ \hat{U}_{10}\otimes\hat{U}_{10},\ \hat{U}_{11}\otimes\hat{U}_{01}\right\}\ .
\end{aligned}$$

The eigenstates to the first set form the Bell basis $|00\rangle \pm |11\rangle$, $|01\rangle \pm |10\rangle$, the second set has $i\,|00\rangle \pm |11\rangle$, $i\,|01\rangle \pm |10\rangle$ as eigenstates and the third gives $|00\rangle + |01\rangle + |10\rangle - |11\rangle$, $|00\rangle + |01\rangle - |10\rangle + |11\rangle$, $|00\rangle - |01\rangle + |10\rangle + |11\rangle$, $-|00\rangle + |01\rangle + |10\rangle + |11\rangle$. The remaining sets have eigenstates similar to the ones given by the third set, except for a phase. All these states are totally mixed (purity factor $p_1 = 0$) locally, and pure in total ($p_2 = 1$). From the table for the cluster sums, one can see that in the case of spin 1/2 particles ($n = 2$) there are no states with more entanglement than the generalized cat states (see next chapter). For $n = 3$-level systems this is also the case for up to 3 particles, but in the case of 4 particles there exists a state with a higher cluster sum $Y_4 = 64$. It turns out that this state is the dyadic product of two Bell states (each with $Y_2 = 8$). This indicates, that a cat state may not necessarily have the largest $Y_N$-value (cf. eq. (5.10)).

We generalize the Bell basis to systems with an arbitrary number of $N$ particles where each particle is a $n$-level subsystem. Our starting point is the state $|Cat\rangle_0 := \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \bigotimes_{k=1}^{N} |i\rangle$ which has the nice property, that any subsystem of $m$ particles has an entanglement of $S(\hat{\rho}_m) = \log_2 n$, when measured in terms of the local von Neumann entropy. ($\hat{\rho}_m$ is the reduced density-operator of the respective $m$-cluster.) So any cluster, regardless of its size $m$, ($m < N$) has the same entropy and furthermore this entropy is the maximum amount of entanglement a $n$-level system can have. In this sense this state can be considered as a maximal entangled state. Since entanglement properties are invariant under local unitary transformations [7], one can generate a complete basis set from $|Cat\rangle_0$ by applying the discrete set of unitary basis operators $\hat{U}_i$. The result,

$$|Cat\rangle_{\mathbf{c}} := \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \omega_n^{j\,c_1}\ |j\rangle \bigotimes_{k=2}^{N} \underline{|j + c_k\rangle}\ , \qquad (5.6)$$

is an explicit definition of the cat state basis. The index $\mathbf{c} = \{c_i\},\ c_i \in [0, n-1]$ labels the $n^N$ states and $\omega_n$ is given by eq. (2.8). Orthonormality and completeness read

$$_{\mathbf{d}}\langle Cat\,|Cat\rangle_{\mathbf{c}} = \delta_{\mathbf{c},\mathbf{d}}\ , \qquad \sum_{\mathbf{c}} |Cat\rangle_{\mathbf{c}\ \mathbf{c}}\,\langle Cat\,| = \hat{1}\ . \quad (5.7)$$

In the low dimensional case of $N = n = 2$ the definition (5.6) reduces to the bell basis and for $N = 3$, $n = 2$ the GHZ state $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ is one member of such a set. Further examples for cat-states are given in Appendix C.

The characterization of these generalized cat states in terms of cluster sums shows (cf. Table I)

$$Y_1 = 0\ , \qquad (5.8)$$

$$Y_m = \frac{(n-1)^m + (-1)^m (n-1)}{n}\ , \qquad 1 \le m < N\ , \qquad (5.9)$$

$$Y_N = (n-1)\,n^{N-1} + \frac{(n-1)^N + (-1)^N (n-1)}{n}\ , \qquad (5.10)$$

while the purity factor distribution for any such a cat state is (cf. eq. (3.8))

$$p_1 = 0\ , \qquad \text{(totaly mixed)} \qquad (5.11)$$

$$p_m = \frac{n^{m-1} - 1}{n^m - 1}\ , \qquad 1 < m < N \qquad (5.12)$$

$$p_N = 1\ , \qquad \text{(pure)}\ . \qquad (5.13)$$

The purity factor distribution for cat-states is shown in Fig. 1.

For $n \gg 1$ any cluster, regardless of its size $m$, ($m < N$) approaches the same purity factor $p_m = \frac{1}{n}$. This behaviour is reminescent of that for the cluster-entropy.

For $N \to \infty$ the cluster-sum $Y_N$ almost exploits the sum rule (3.7):

$$\frac{Y_N}{n^N} \to \frac{n-1}{n} \qquad (5.14)$$

Cat states thus become even more "non-classical" with increasing $N$ (and $n$).

### B. Collective invariants and eigenstates

We have already briefly discussed invariants under general unitary transformations ($\mathrm{tr}\left\{\hat{\rho}^2\right\}$) and under local

unitary transformations (the cluster sums). Let our quantum network be described by the Hamiltonian $\hat{H}$, with the spectral representation

$$\hat{H} = \sum_k E_k \hat{\rho}_k \tag{5.15}$$

where $\hat{\rho}_k = |\phi_k\rangle \langle\phi_k|$. Then the expectation-values

$$C_j = \mathrm{tr}\,\{\hat{\rho}\hat{\rho}_j\} = \langle\phi_j|\,\hat{\rho}\,|\phi_j\rangle \tag{5.16}$$

constrained by $\sum_j C_j = 1$ are additional invariants of motion under the unitary transformation generated by $\hat{H}$. This follows from the fact, that $i\hbar\frac{\partial}{\partial t}\,\mathrm{tr}\left\{\hat{\rho}\hat{A}\right\} = \mathrm{tr}\left\{[\hat{H},\hat{\rho}]_-\hat{A}\right\} = \mathrm{tr}\left\{[\hat{A},\hat{H}]_-\hat{\rho}\right\}$ which vanishes if $\hat{A}$ commutes with $\hat{H}$. Note that $|\phi_j\rangle$ has, in general, not product form. We consider 3 examples:

i. $N = n = 2$ network with Förster-interaction $C_\mathrm{F}$

The Hamiltonian in terms of the collective operators (cf. Sect. IV A)

$$\hat{H} = \frac{\hbar\,\omega}{2}\,\hat{E}_{001,0} + \frac{\hbar\,C_\mathrm{F}}{2}\,(\hat{E}_{200,0} + \hat{E}_{020,0}) \tag{5.17}$$

has the eigenstates

$$|\phi_0\rangle = |00\rangle \tag{5.18}$$
$$|\phi_\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) \tag{5.19}$$
$$|\phi_1\rangle = |11\rangle \tag{5.20}$$

which can be rewritten as

$$\hat{\rho}_0 = \frac{1}{4}(\hat{1} + \hat{E}_{002,0} - \hat{E}_{001,0}) \tag{5.21}$$
$$\hat{\rho}_\pm = \frac{1}{4}(\hat{1} - \hat{E}_{002,0} \pm (\hat{E}_{200,0} + \hat{E}_{020,0})) \tag{5.22}$$
$$\hat{\rho}_1 = \frac{1}{4}(\hat{1} + \hat{E}_{002,0} + \hat{E}_{001,0}) \tag{5.23}$$

These are all permutation-symmetric density-operators; their expectation values $C_j$ according to eq. (5.16) lead to three independent collective invariants

$$E_{001,0} = \text{const.} \tag{5.24}$$
$$E_{002,0} = \text{const.} \tag{5.25}$$
$$E_{200,0} + E_{020,0} = \text{const.} \tag{5.26}$$

ii. $N = n = 2$ with renormalization interaction $C_\mathrm{R}$

The Hamiltonian

$$\hat{H} = \frac{\hbar}{4}\,(\omega^{(1)} + \omega^{(2)})\hat{E}_{001,0} + \frac{\hbar}{4}\,(\omega^{(1)} - \omega^{(2)})\hat{E}_{001,1} + \frac{\hbar\,C_\mathrm{R}}{2}\hat{E}_{002,0} \tag{5.27}$$

has the product states as eigenstates. The collective invariants are in this case

$$E_{001,0} = \text{const.} \tag{5.28}$$
$$E_{002,0} = \text{const.} \tag{5.29}$$
$$E_{001,1} = \text{const.} \tag{5.30}$$

iii. $N = n = 2$ with collective stimulation

The Hamiltonian given by

$$\hat{H} = \frac{\hbar}{2}\,g\;\hat{E}_{100,0} + \frac{\hbar}{2}\,\delta\;\hat{E}_{001,0} \tag{5.31}$$

describes a non-interacting spin-pair driven by a coherent electromagnetic field (strength $g$, detuning $\delta$) in rotating wave approximation. Its invariants turn out to be

$$\delta E_{001,0} + g E_{100,0} = \text{const.} \tag{5.32}$$
$$\delta^2 E_{002,0} + 2g\delta E_{101,0} + g^2 E_{200,0} = \text{const.} \tag{5.33}$$
$$4\delta^2(g^2+\delta^2)E_{001,1} - g^4 E_{002,0} - g^2(g^2+\delta^2)E_{020,0} + 4g\delta(g^2+\delta^2)E_{100,1} + 2g^3\delta E_{101,0} - g^2\delta^2 E_{200,0} = \text{const.} \tag{5.34}$$

Contrary to example ii, the $\hat{H}$-model (5.31) is permutation-symmetric, while the invariant (5.34) contains expectation values with $b \neq 0$. The latter can be unequal zero only if at some previous stage of the preparation the permutation symmetry had been broken (cf. Sect. V E 4).

### C. Permutation symmetry

A complete description of large quantum networks turns out to be virtually impossible, as the Hilbert space dimension grows exponentially with the number of particles used. However, symmetry can reduce the complexity of the total system by a significant amount: For a perfect permutation-symmetric system we will show that the reduction in the number of parameters needed is so enormous, that a polynomial increasing number of parameters is enough to describe networks of arbitrary size.

Within such a network no subsystem can be distinguished, neither in preparation nor in detection. Two possible setups can be thought of: Fundamental indistinguishable and operational indistinguishable subsystems.

A typical setup of the former type would be several electrons in a box. As the electrons are fermions and their location is not fixed, there is no way to act independently on a specific electron. All controlling and measurement procedures act on the system as a whole. Therefore no information loss can occur when reducing the system-description to permutation-symmetric operators.

Opposed to that, in the case of operational indistinguishability, the design of the experimental setup is the source of reduction. A linear ion trap with a laser beam acting on the ions could be used as an example. Only if the beam waist is less than the spatial separation of the ions, the particles become distinguishable, so by controlling the laser beam the experimentalist can choose which case he is in.

Collective operators $\hat{E}_{\alpha\beta\gamma,b}$ with $b = 0$ (cf. Sect. IV A) are invariant under any arbitrary subsystem permutation $\hat{P}$

$$\hat{P}^\dagger \hat{E}_{\alpha\beta\gamma,0} \hat{P} = \hat{E}_{\alpha\beta\gamma,0} \ . \tag{5.35}$$

For indistinguishable subsystems, these are the only operators allowed: Any control or measurement operator has to be part of the Liouville-space spanned by $\hat{E}_{\alpha\beta\gamma,0}$. The number of such operators is (for $n = 2$)

$$\xi_0 = \frac{(N+1)(N+2)(N+3)}{6} \ , \tag{5.36}$$

and thus scales polynomially of order $\mathcal{O}(N^3)$ with the number of subsystems involved. Therefore, such a highly symmetric system constitutes a class of reduced complexity as compared with a general quantum network of the same size $N$. This is somewhat surprising as indistinguishability might have been expected to enhance non-classical features.

Alternatively, if all expectation-values happen to be permutation-symmetric (no "structure"), then

$$\langle \hat{C}_{\alpha\beta\gamma,p} \rangle = \langle \hat{C}_{\alpha\beta\gamma} \rangle \tag{5.37}$$

for all $p$-permutations, and by recalling eq. (2.9) the definition eq. (4.2) leads to

$$\langle \hat{E}_{\alpha\beta\gamma,b} \rangle = \Omega \langle \hat{C}_{\alpha\beta\gamma} \rangle \delta_{0b} \tag{5.38}$$

i.e. we need to consider only the $\hat{E}_{\alpha\beta\gamma,0}$ operators. This would hold, correspondingly, for the operators $\hat{F}_{z,\gamma,b}$ according to eq. (4.7) (or $\hat{G}_{m,b}$ according to eq. (4.8)) provided only $z$, $\gamma$ (or $m$) could indeed be distinguished. The number of remaining operators is then further reduced: For $\hat{F}_{z,\gamma,0}$ this number is $\sum_{\gamma=0}^{N}(2(N-\gamma)+1) = (N+1)^2$, for $\hat{G}_{m,0}$ it is simply $N+1$.

#### 1. Symmetry classes

Permutation symmetry defines a kind of operational indistinguishability between the subsystems of a given network. This symmetry alone, however, would allow for more symmetry classes than realized in nature by the fundamentally indistinguishable particles, Fermions and Bosons, respectively. The spin-statistics-relation going back to Pauli [11,13] might be relaxed, if the particles or subsystems are localized in different areas of real space [12]. The location index would, in principle, render these subsystems distinguishable; however, for the following we assume that the actual operators describing the network and its coupling to the outside world are still permutation-symmetric, so that the corresponding super-selection rules apply, as will be discussed below. This would imply, e.g. that electrons, localized in different semiconductor quantum dots, could live in the state-subspace of Bose-symmetry (or of any other "paraboson" [13] symmetry class as well; cf. Table III), provided we are able to prepare such states from some standard initial state: Directed transient symmetry breaking could do this job (cf. Sect. V E 4).

Any symmetry type can be characterized by a Young diagram and is equivalent to a irreducible representation of the permutation group $S_N$ spanned by the basis vectors. As an example, Table III shows all possible symmetries for a $N = 4$ particle system. Since a definite angular momentum quantum number $j$ is assigned to every Young diagram, there are $2j+1$ states of equal symmetry type but different energy (i.e. different configuration). If the system is subjected to permutation-symmetric operators only, the super-selection rules prohibit any transition between different Young tables. In case the system under consideration is in a specific state with angular momentum $j$, the state space reached by applying any collective operator $\hat{E}_{ijk,0}$ is of dimension $2j+1$. Therefore $(2j+1)^2$ parameters (expectation values) are needed to describe the system. The total number of parameters, $\sum_{j=0}^{N/2}(2j+1)^2 = \frac{1}{6}(N+1)(N+2)(N+3)$ is exactly the number $\xi_0$ of collective operators $\hat{E}_{ijk,0}$ (see eq. (5.36)).

#### 2. Structure and Hamilton-models

For fundamentally indistinguishable particles, the subsystem index $\mu$ has no physical meaning. However, it may happen that a specific property is not only a good quantum number but a unique constant of motion for any subsystem. Localization in real space is a pertinent example; the subsystem-index $\mu$ is then mapped onto a spatial position-index $R_\mu$. By this position any subsystem becomes distinguishable, in principle. The phases (entering the collective operators $\hat{E}$) get a physical meaning ("wave length"). An "operational" indistinguishability remains, if the Hamiltonian describing the network does still contain collective-operators of the type $\hat{E}_{\alpha\beta\gamma,0}$ only.

Typical Hamilton-models include $m = 1, 2$ -particle operators. The structure tends to break permutation-symmetry for the localized states as the coupling usually depends on the distance $|R_\mu - R_\nu|$. This partial selec-

tivity can be described as a perturbation via collective operators $\hat{E}_{\alpha\beta\gamma,b}$ with $b \neq 0$.

If all pertinent distances could be made equal, the breaking of the permutation symmetry would go to zero. For $N > D+1$ (in $D$ dimensions) however, the interaction distances cannot all be the same. A partial remedy is the introduction of a "quantum bus": In this case the nodes do not interact directly but only indirectly via a common degree of freedom. This degree of freedom could be a central spin, but typically is implemented as a collective mode (like the phonon mode of a cold ion trap [14]).

### D. Symmetry breaking and irreversibility

Constrained operations will lead, quantum mechanically, to selection rules accompanied by a tremendous reduction of the state space available to the system dynamics starting from a given initial state. This situation needs to be distinguished from lack of control (measurement data) implying lack of information (entropy $S > 0$). In so far as this lack of control refers to expectation values $b \neq 0$ (which would be absent for strict permutation symmetry), "uncontrolled" symmetry breaking may be said to lead to an ensemble description and irreversibility: The true state space has been reduced to the smaller one defined by the assumed permutation symmetry. In this case, however, the "decoherence" does not reflect the influence of an external bath but is rather of "internal" origin. Phenomenologically ony may try to model these effects, as usual, via some decoherence times; however, it is not clear yet under what conditions such a procedure would be appropriate. Clearly it should, at most, work for sufficiently large networks and only to the extent that the reduction is really substantial (cf. ref. [15]). The decoherence time would then, in turn, allow to assess symmetry breaking effects in a global way.

In the following, we will restrict ourselves to unitary transformations.

### E. Special unitary transformations

#### *1. Cyclic permutations: generalized echoes*

The pulse-like manipulation of Hamiltonians,

$$\hat{H}(t) = \hat{H}_j \ , \qquad \text{for } t_j \leq t < t_{j+1} \ , \quad j = 0, 1, 2, \ldots \tag{5.39}$$

to shape unitary evolutions

$$\hat{U}_j(\Delta t_j) = e^{\frac{-\hat{H}_j \Delta t_j}{\hbar}} \ , \qquad \Delta t_j = t_{j+1} - t_j \tag{5.40}$$

at will, has become a popular approach to quantum computation [3]. One basic operation is the "halting operation", $\hat{U}_j(\Delta t_j) = \hat{1}$, which would require $\hat{H}_j \equiv 0$. This condition cannot be realized, in general. The well-known spin-echoes [17] are formally based on time reversal, thus undoing a unitary evolution over a time period $\Delta t/2$ within the next period $\Delta t/2$:

$$\hat{U}_j(-\Delta t/2) = \hat{U}_j^{\dagger}(\Delta t/2) \tag{5.41}$$

This time-reversal cannot be implemented either; however, its effect can be simulated in any discrete Hilbert-space of dimension $n$ by means of the cyclic permutation operations $\hat{U}_{n-1,0}$, as introduced in section II B.

Let us consider the time-independent Hamiltonian $\hat{H}$. It is convenient to require $\mathrm{tr}\left\{\hat{H}\right\} = 0$ so that

$$\sum_k^{n-1} E_k = 0 \tag{5.42}$$

with $E_k$ denoting the eigenvalues of $\hat{H}$. Then the effect of the unitary time evolution generated by $\hat{H}$

$$|\psi(\Delta t)\rangle = \hat{U}_{\hat{H}}(\Delta t)\,|\psi(0)\rangle \tag{5.43}$$

can be suppressed for any given $\Delta t$ by the iteration

$$\hat{U}_{\mathrm{eff}}(\Delta t) := \left(\hat{U}_{n-1,0}\hat{U}_{\hat{H}}(\Delta t/n)\right)^n \tag{5.44}$$

Here we have assumed for simplicity that the application of $\hat{U}_{n-1,0}$ (cf. eq. (2.8)) does not consume any additional time. $\hat{U}_{n-1,0}$ generates cyclic permutations between all the states $E_k$ of the spectrum. Any initial eigenstate thus visits all the other eigenstates for the same time $\Delta t/n$ so that, due to the constraint eq. (5.42) the total acquired phase adds up to zero. This invariance property holds for any initial state $|\psi(0)\rangle$, as such a state can be written as a superposition of eigenstates, and any eigenstate returns to its initial phase separately.

This mapping, eq. (5.44), can be repeated to get the stroboscopic invariance (cf. ref. [16])

$$\hat{U}_{\mathrm{eff}}(m\Delta t) = \left(\hat{U}_{n-1,0}\hat{U}_0(\Delta t/n)\right)^{nm} = \hat{1} \ , \tag{5.45}$$

where any initial state is periodically recovered. The cyclic permutation $\hat{U}_{n-1,0}$ can be decomposed into a sequence of $n-1$ state-selective $\pi$-pulses; for $n = 3$, e.g.:

$$\begin{aligned} \hat{U}_{2,0} &= \hat{P}_{01} + \hat{P}_{12} + \hat{P}_{20} \\ &= (\hat{P}_{00} + \hat{P}_{12} + \hat{P}_{21}) \cdot (\hat{P}_{01} + \hat{P}_{10} + \hat{P}_{22}) \end{aligned} \tag{5.46}$$

The total number of such $\pi$-pulses for $\hat{U}_{eff}(m\Delta t)$ is $n(n-1)m$. The conventional spin-echo obtains for $n = 2$, $m = 1$, and an initial state $|\psi(0)\rangle$ generated from the ground state by means of a $\pi/2$-pulse. The second $\hat{U}_{1,0}$-flip is then usually omitted.

*2. Selective control*

In principle, the results obtained in Sect. V E 1 may also be used for a network of $N$ subsystems of dimension $n$ each. The condition is, that one is able to implement the respective cyclic operator: For a network with (distinguishable) non-interacting subsystems we simply have to replace $\hat{U}_{n-1,0}$ by the product-operator $\hat{Q} = \hat{U}^{(1)}_{n-1,0} \otimes \ldots \otimes \hat{U}^{(N)}_{n-1,0}$.

In the case of identical subsystems, interactions may lift the degeneracy of the network-eigenstates. We consider the example of a network of 2-level-subsystems and an interaction of the type

$$\hat{H}' = \hbar \sum_{\mu<\nu} C_R^{\mu\nu}\, \hat{\lambda}_3^{(\mu)} \otimes \hat{\lambda}_3^{(\nu)} \; . \tag{5.47}$$

This interaction modifies the eigen-spectrum (allowing for selectivity) while leaving the eigenstates unchanged (i.e. as product-states), $|p(N)\ldots p(1)\rangle$; $p(\mu) = 0, 1$ (cf. eq. (5.27)). Single-particle transitions are allowed only between states which differ at one position ("Hamming distance" 1). A convenient numbering, $s$, of the $2^N$ states allowing for cyclic permutation is given in Table II. For a large network $N$, the echo becomes difficult to implement practically, as it requires $\sim 2^{2N}$ short but at the same time frequency-selective $\pi$-pulses per time period $\Delta t$. In general, interactions will also modify the eigenstates (i.e. lead to a non-product form). A simple example for $N = n = 2$ is provided by eq. (5.18) - (5.20).

It has been proposed [18] to exploit some approximate stroboscopic invariance also for open systems under the condition $\Delta t \ll \tau_c$, where $\tau_c$ is a typical time-scale for dissipation. However, such short pulses will easily come in conflict with their required selectivity in frequency-space and thus tend to restrict practical implementations of their proposal to small networks only. Furthermore, it is unlikely that the additional control interactions could work without inducing uncontrollable features on their own. Supression of decoherence would amount to increase the pertinent decoherence-time and -length to macroscopic dimensions (which typically happens for very special and restricted state-spaces only, like in superconductivity or Bose-Einstein condensation).

| $N\backslash s$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ... |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 1 | | | | | | | | |
| 2 | 00 | 01 | 11 | 10 | | | | | | |
| 3 | 000 | 001 | 011 | 010 | 110 | 111 | 101 | 100 | | |
| 4 | 0000 | 0001 | 0011 | 0010 | 0110 | 0111 | 0101 | 0100 | 1100 | ... |

TABLE II. Circular sequences of product states with Hamming distance 1 for various network sizes $N$. These sequences can be constructed recursively: Sequence $N+1$ is obtained from that of $N$ by repeating each member and supplementing it from the right by $0, 1$ $(1, 0)$, respectively. (There are other sequences with the same property.)

*3. Collective control*

We consider the unitary evolution induced by a "collective" and permutation-symmmetric Hamiltonian like

$$\hat{H} = \hbar\, \alpha\, \hat{E}_{m00,0} = \hbar\, \alpha \sum_{\mu<\nu\ldots<\xi} \hat{\sigma}_x^{(\mu)} \hat{\sigma}_x^{(\nu)} \ldots \hat{\sigma}_x^{(\xi)} \tag{5.48}$$

This $\hat{H}$ generates the unitary time evolution

$$\begin{aligned} \hat{U}(t) &= e^{-i\alpha t \hat{E}_{m00,0}} \\ &= \prod_{\mu<\nu\ldots<\xi} \left(\cos(\alpha t)\, \hat{1} - i\, \sin(\alpha t)\, \hat{\sigma}_x^{(\mu)} \hat{\sigma}_x^{(\nu)} \ldots \hat{\sigma}_x^{(\xi)}\right) . \end{aligned} \tag{5.49}$$

In particular for $m = 1$ and $\alpha t = \frac{\pi}{2}$ on finds $\hat{U}_{\frac{\pi}{2}} = (-i)^N \hat{E}_{N00,0}$, while for $\alpha t = \frac{\pi}{4}$, $\hat{U}_{\frac{\pi}{4}} \propto \sum_{\mu=0}^{N} (-i)^\mu \hat{E}_{\mu 00,0}$.

For $m = 2$ and $\alpha t = \frac{\pi}{2}$, we have $\hat{U}_{\frac{\pi}{2}} = (-i)^{\frac{N}{2}} \hat{E}_{N00,0}$ for $N$ even and $\hat{U}_{\frac{\pi}{2}} = i^{\frac{N-1}{2}} \hat{1}$ otherwise; for $N$ even and time $\alpha t = \frac{\pi}{4}$, eq. (5.49) simplifies to

$$\hat{U}_{\frac{\pi}{4}} \propto \hat{1} + i \hat{E}_{N00,0} \; , \quad \text{for } \frac{N}{2} \text{ even} \tag{5.50}$$

$$\hat{U}_{\frac{\pi}{4}} \propto \hat{1} - i \hat{E}_{N00,0} \; , \quad \text{for } \frac{N}{2} \text{ odd} \tag{5.51}$$

Applied to the ground state $|00\ldots0\rangle$, this $\hat{U}_{\frac{\pi}{4}}$ thus creates the $N$-particle cat state (cf. Sect. V A 2 and Appendix C)

$$\hat{U}_{\frac{\pi}{4}} |00\ldots0\rangle \propto |00\ldots0\rangle \pm i\, |11\ldots1\rangle \tag{5.52}$$

in a single step! Specific examples for such collective control scenarios have been discussed in Refs. [19–24].

*4. Interplay between selective and collective interactions*

Selective interactions violate the selection rules implicit in permutation-symmetric interactions. This qualitatively different dynamical behaviour can be exploited to implement specific functionalities: One possibility is to address different symmetry-classes via selective coupling (i.e. controlled symmetry breaking); another possibility is to suppress transition due to permutation-symmetric interactions by using states of different symmetry, which, nevertheless, could all be prepared selectively ("decoherence-free subspace").

To be specific, let us consider a network of $N$ pseudo-spins without mutual interactions, but in the presence of a quantum bus (i.e. a collective mode to which all spins are coupled in the same way). We further assume that the coupling to a larger field can be made at will either selective or collective. We start (for $N = 4$, $n = 2$) with the permutation-symmetric ground state $|0000\rangle$, $(j = 2)$.

By applying selective laser pulses and exploiting the coupling to the quantum-bus we can generate the EPR-state $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ within the pair $\nu = 1, 2$. The total 4-particle state then becomes a member of the symmetry class $j = 1$, see Table III. If permutation symmetry is restored now, transitions are possible only within the $(2j+1)$-dimensional subspace of this symmetry class (on a time-scale less than the decoherence time). This would allow to study finite systems of (operationally) indistinguishable subsystems of a symmetry not realized in nature by fundamentally indistinguishable particles! It is straightforward to extend this scheme to $N > 4$.

Alternatively, the selection rules can readily be exploited as a means for stabilization: For this purpose we assume that the "unwanted" dynamics (coupling to the bath) is permutation-symmetric, while the control dynamics to be used is selective (i.e. not subject to the selection rules). For $N = 4$ we could take the three lowest energy states $j = 2, 1, 0$. Unitary dynamics within this subspace would (in the ideal case) not be perturbed by dissipation, if the bath was kept at zero temperature. (There are higher energy levels with the same symmetry, though.)

It is thus preferable to use the multiplicity of the states $j = 0$ (for $N$ even) which would decouple from the bath exactly, as each of those states is the only member of its symmetry class (see Table III): There is nothing to connect to under the action of a permutation-symmetric coupling (cf. ref [25]). Such schemes have been investigated by a number of authors [26–30]. The resulting stabilization is limited by the fact that the symmetry selection rules will, in practice, not hold strictly.

## VI. SUMMARY AND CONCLUSIONS

In this paper we have addressed composite systems consisting of $N$ subsystems $\mu$ of respective dimension $n_\mu$. We have been concerned with applications of (selective) cluster-operators and collective operators, both based on unitary local operator sets, and both being orthogonal and complete. The former are very useful in characterizing entanglement of pure network states, in particular of cat-states, for any $n$ and $N$. Commuting sets of cluster-operators have been derived. The latter are adapted to situations in which individual subsystems cannot be selected. Classically this would imply a loss of information about the resulting state. Quantum-mechanically, this lack of control would rather give rise to entanglement – with super-selection rules tremendously reducing the state-space available: Actually its dimension becomes polynomial in $N$. Such states have been proposed as subspaces of reduced decoherence (with respect to permutation-symmetric bath interations). We have suggested that localized, artificial quantum networks, subjected to certain symmetry-breaking initialization steps, might live in state-symmetry-classes that otherwise cannot occur for fundamentally indistinguishable particles.


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## APPENDIX A: UNITARY OPERATOR BASIS FOR $N = 3$

$$\hat{U}_{00} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \hat{U}_{01} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{2\pi i}{3}} & 0 \\ 0 & 0 & e^{-\frac{2\pi i}{3}} \end{pmatrix} \quad \hat{U}_{02} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\frac{2\pi i}{3}} & 0 \\ 0 & 0 & e^{\frac{2\pi i}{3}} \end{pmatrix}$$

$$\hat{U}_{10} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \hat{U}_{11} = \begin{pmatrix} 0 & 0 & e^{-\frac{2\pi i}{3}} \\ 1 & 0 & 0 \\ 0 & e^{\frac{2\pi i}{3}} & 0 \end{pmatrix} \quad \hat{U}_{12} = \begin{pmatrix} 0 & 0 & e^{\frac{2\pi i}{3}} \\ 1 & 0 & 0 \\ 0 & e^{-\frac{2\pi i}{3}} & 0 \end{pmatrix}$$

$$\hat{U}_{20} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \hat{U}_{21} = \begin{pmatrix} 0 & e^{\frac{2\pi i}{3}} & 0 \\ 0 & 0 & e^{-\frac{2\pi i}{3}} \\ 1 & 0 & 0 \end{pmatrix} \quad \hat{U}_{22} = \begin{pmatrix} 0 & e^{-\frac{2\pi i}{3}} & 0 \\ 0 & 0 & e^{\frac{2\pi i}{3}} \\ 1 & 0 & 0 \end{pmatrix}$$

## APPENDIX B: RELATIONS BETWEEN OPERATOR SETS

The relations between this unitary operator set $\left\{\hat{U}_{ab}\right\}$, the SU($n$) generators $\{\hat{u}_{jk}, \hat{v}_{jk}, \hat{w}_l\}$ and the projection operators $\left\{\hat{P}_{ij}\right\}$ are given by:

$$\hat{P}_{jk} = \frac{1}{n} \sum_{b=0}^{n-1} \omega_n^{-bk} \hat{U}_{\underline{j-k},b} \tag{B1}$$

$$\hat{U}_{ab} = \sum_{k=0}^{n-1} \omega_n^{kb} \hat{P}_{\underline{a+k},k} \tag{B2}$$

$$\begin{aligned} \hat{U}_{ab} = & \sum_{0 \le j < k < n} \frac{1}{2} (\omega_n^{bj} \delta_{\underline{j+a},k} + \omega_n^{bk} \delta_{\underline{k+a},j})\, \hat{u}_{jk} + \\ & \sum_{0 \le j < k < n} \frac{i}{2} (\omega_n^{bj} \delta_{\underline{j+a},k} - \omega_n^{bk} \delta_{\underline{k+a},j})\, \hat{v}_{jk} + \\ & \sum_{0 \le l < n-1} -\frac{1}{\sqrt{2(l+1)(l+2)}}\, \delta_{0,a} \left(-(l+1)\, \omega_n^{b(l+1)} + \sum_{q=1}^{l+1} \omega_n^{b(q-1)}\right) \hat{w}_l \end{aligned} \tag{B3}$$

$$\begin{aligned} \hat{u}_{jk} &= \frac{1}{n} \sum_{a,b=0}^{n-1} (\omega_n^{-bk}\, \delta_{a,\underline{j-k}} + \omega_n^{-bj}\, \delta_{a,k-j})\, \hat{U}_{ab} \\ \hat{v}_{jk} &= \frac{i}{n} \sum_{a,b=0}^{n-1} (\omega_n^{-bk}\, \delta_{a,\underline{j-k}} - \omega_n^{-bj}\, \delta_{a,k-j})\, \hat{U}_{ab} \\ \hat{w}_l &= -\frac{1}{n} \sqrt{\frac{2}{(l+1)(l+2)}} \sum_{b=0}^{n-1} (-(l+1)\, \omega_n^{-b(l+1)} + \sum_{q=1}^{l+1} \omega_n^{-b(q-1)})\, \hat{U}_{0,b} \end{aligned} \tag{B4}$$

## APPENDIX C: EXAMPLES FOR GENERALIZED CAT STATES

We give some (low dimensional) examples for generalized cat states consisting of $N$ subsystems with each one being a $n$ level system. Together they form a orthonormal and complete basis of the underlying $n^N$ dimensional Hilbert space.

$N = 2$:

$n = 2$:

$$|Cat\rangle_{00} = \tfrac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$
$$|Cat\rangle_{01} = \tfrac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$$
$$|Cat\rangle_{10} = \tfrac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)$$
$$|Cat\rangle_{11} = \tfrac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

$n = 3$:

$$|Cat\rangle_{00} = \tfrac{1}{\sqrt{3}}\left(|00\rangle + |11\rangle + |22\rangle\right)$$
$$|Cat\rangle_{01} = \tfrac{1}{\sqrt{3}}\left(|01\rangle + |12\rangle + |20\rangle\right)$$
$$|Cat\rangle_{02} = \tfrac{1}{\sqrt{3}}\left(|02\rangle + |10\rangle + |21\rangle\right)$$

$$|Cat\rangle_{10} = \tfrac{1}{\sqrt{3}}\left(|00\rangle + e^{\frac{2}{3}\pi i}|11\rangle + e^{-\frac{2}{3}\pi i}|22\rangle\right)$$
$$|Cat\rangle_{11} = \tfrac{1}{\sqrt{3}}\left(|01\rangle + e^{\frac{2}{3}\pi i}|12\rangle + e^{-\frac{2}{3}\pi i}|20\rangle\right)$$
$$|Cat\rangle_{12} = \tfrac{1}{\sqrt{3}}\left(|02\rangle + e^{\frac{2}{3}\pi i}|10\rangle + e^{-\frac{2}{3}\pi i}|21\rangle\right)$$
$$|Cat\rangle_{20} = \tfrac{1}{\sqrt{3}}\left(|00\rangle + e^{-\frac{2}{3}\pi i}|11\rangle + e^{\frac{2}{3}\pi i}|22\rangle\right)$$
$$|Cat\rangle_{21} = \tfrac{1}{\sqrt{3}}\left(|01\rangle + e^{-\frac{2}{3}\pi i}|12\rangle + e^{\frac{2}{3}\pi i}|20\rangle\right)$$
$$|Cat\rangle_{22} = \tfrac{1}{\sqrt{3}}\left(|02\rangle + e^{-\frac{2}{3}\pi i}|10\rangle + e^{\frac{2}{3}\pi i}|21\rangle\right)$$

$N = 3$:

$n = 2$:

$$|Cat\rangle_{000} = \tfrac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$
$$|Cat\rangle_{001} = \tfrac{1}{\sqrt{2}}\left(|001\rangle + |110\rangle\right)$$
$$|Cat\rangle_{010} = \tfrac{1}{\sqrt{2}}\left(|010\rangle + |101\rangle\right)$$
$$|Cat\rangle_{011} = \tfrac{1}{\sqrt{2}}\left(|011\rangle + |100\rangle\right)$$
$$|Cat\rangle_{100} = \tfrac{1}{\sqrt{2}}\left(|000\rangle - |111\rangle\right)$$
$$|Cat\rangle_{101} = \tfrac{1}{\sqrt{2}}\left(|001\rangle - |110\rangle\right)$$
$$|Cat\rangle_{110} = \tfrac{1}{\sqrt{2}}\left(|010\rangle - |101\rangle\right)$$
$$|Cat\rangle_{111} = \tfrac{1}{\sqrt{2}}\left(|011\rangle - |100\rangle\right)$$

The cat-states $N = n = 2$ can entirely be expressed in terms of 2-particle collective operators:

$$\hat{\rho}_{00} = \frac{1}{4}(\hat{1} + \hat{E}_{200,0} - \hat{E}_{020,0} + \hat{E}_{002,0}) \tag{C1}$$
$$\hat{\rho}_{01} = \frac{1}{4}(\hat{1} + \hat{E}_{200,0} + \hat{E}_{020,0} - \hat{E}_{002,0}) \tag{C2}$$
$$\hat{\rho}_{10} = \frac{1}{4}(\hat{1} - \hat{E}_{200,0} + \hat{E}_{020,0} + \hat{E}_{002,0}) \tag{C3}$$
$$\hat{\rho}_{11} = \frac{1}{4}(\hat{1} - \hat{E}_{200,0} - \hat{E}_{020,0} - \hat{E}_{002,0}) \tag{C4}$$

These are all permutation-symmetric and involve $N =$ 2-particle operators only. For $N > n = 2$ collective operators appear also of order $< N$: For $N = 3$, $|\mathrm{Cat}\rangle_{000} = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, e.g., one finds

$$\hat{\rho}_{000} = \frac{1}{2^3}(\hat{1} + \hat{E}_{300,0} - \hat{E}_{120,0} + \hat{E}_{002,0}) \tag{C5}$$

The last term is responsible for the surviving 2-particle properties. In general, the cat-states are no longer permutation-symmetric: A pertinent example is

$$\begin{aligned}\hat{\rho}_{001} =& \frac{1}{2^3}(\hat{1} - \frac{1}{3}\hat{E}_{002,0} + \frac{2}{3}\hat{E}_{002,1} + \frac{2}{3}\hat{E}_{002,2} \\ &+ \frac{1}{3}\hat{E}_{120,0} + \frac{2}{3}e^{-i\frac{\pi}{3}}\hat{E}_{120,1} + \frac{2}{3}e^{-i\frac{\pi}{3}}\hat{E}_{120,2})\ .\end{aligned} \tag{C6}$$


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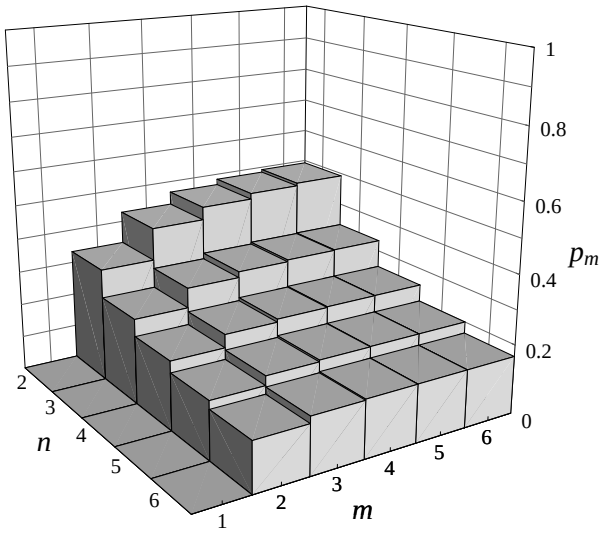


FIG. 1. $m$-cluster purity factor $p_m$, as defined in eq. (3.8), for cat-states depending on the subsystem dimension $n$ and the cluster size $m$ for $m < N$. The total cat-state is a pure state, so $p_N$ always equals 1.

| Conf. | $m$ | $j$ | Young-T. | Basis vector |
|---|---|---|---|---|
| 1111 | 2 | 2 | 1 2 3 4 | $\|1111\rangle$ |
| 0111 | 1 | 2 | 1 2 3 4 | $\frac{1}{2}(\|1110\rangle + \|1101\rangle + \|1011\rangle + \|0111\rangle)$ |
| | | 1 | 1 2 3<br>4 | $\frac{1}{2\sqrt{3}}(3\,\|1110\rangle - \|1101\rangle - \|1011\rangle - \|0111\rangle)$ |
| | | | 1 2 4<br>3 | $\frac{1}{\sqrt{6}}(2\,\|1101\rangle - \|1011\rangle - \|0111\rangle)$ |
| | | | 1 3 4<br>2 | $\frac{1}{\sqrt{2}}(\|0111\rangle - \|1011\rangle)$ |
| 0011 | 0 | 2 | 1 2 3 4 | $\frac{1}{\sqrt{6}}(\|0011\rangle + \|0101\rangle + \|0110\rangle + \|1001\rangle + \|1010\rangle + \|1100\rangle)$ |
| | | 1 | 1 2 3<br>4 | $\frac{1}{\sqrt{6}}(\|0011\rangle + \|0101\rangle - \|0110\rangle + \|1001\rangle - \|1010\rangle - \|1100\rangle)$ |
| | | | 1 2 4<br>3 | $\frac{1}{2\sqrt{3}}(2\,\|0011\rangle - \|0101\rangle + \|0110\rangle - \|1001\rangle + \|1010\rangle - 2\,\|1100\rangle)$ |
| | | | 1 3 4<br>2 | $\frac{1}{2}(\|0101\rangle + \|0110\rangle - \|1001\rangle - \|1010\rangle)$ |
| | | 0 | 1 2<br>3 4 | $\frac{1}{2}(\|0011\rangle - \|0110\rangle - \|1001\rangle + \|1100\rangle)$ |
| | | | 1 3<br>2 4 | $\frac{1}{2}(\|0101\rangle - \|0110\rangle - \|1001\rangle + \|1010\rangle)$ |
| 0001 | -1 | 2 | 1 2 3 4 | $\frac{1}{2}(\|0001\rangle + \|0010\rangle + \|0100\rangle + \|1000\rangle)$ |
| | | 1 | 1 2 3<br>4 | $\frac{1}{2\sqrt{3}}(3\,\|0001\rangle - \|0010\rangle - \|0100\rangle - \|1000\rangle)$ |
| | | | 1 2 4<br>3 | $\frac{1}{\sqrt{6}}(2\,\|0010\rangle - \|0100\rangle - \|1000\rangle)$ |
| | | | 1 3 4<br>2 | $\frac{1}{\sqrt{2}}(\|1000\rangle - \|0100\rangle)$ |
| 0000 | -2 | 2 | 1 2 3 4 | $\|0000\rangle$ |

TABLE III. Symmetry classes for $N = 4$ subsystems. The configuration reflects the number of spins up or down and therefore the energy level $m$. The angular momentum quantum number $j$ defines the symmetry class with respect to permutation. The basis vectors are uniquely labeled by their Young tableau together with the magnetic quantum number $m$. For permutation-symmetric operators, super-selection rules only allow transitions between states with the same Young tableau.